ON THE PROBLEM OF SYMMETRICAL FLOW PAST A GIVEN SYMMETRICAL PROFILE WITH SUBSONIC VELOCITY AT INFINITY AND LOCAL SUPERSONIC VELOCITIES

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In his paper [1] the author formulated the problem of flow past of profile with the occurrence of local supersonic zones terminated by straight compression shocks; in his paper [2] he formulated the similar problem for the case of curved compression shocks.

For this it was necessary to give the hodograph of a part of the profile and also the hodograph of the compression shock itself. The profile is determined by the latter condition, so that this problem is not direct, but inverse.

In the present paper we show how we can construct, to the first approximation, the flow past an arbitrary given smooth profile with a given velocity at infinity, assuming that we have already solved the problem of paper [1] for a case giving a profile close to that specified in the data of our problem.* The first problem will be denoted for brevity by *I*, and the second by *II*.

It is sufficient to consider the upper half of the flow - above the zero streamline, which consists of two infinite half-lines, lying on the x-axis, and the upper portion of the profile. The upper portion of the profile obtainedfrom the solution of Problem *I* will be denoted by L^0, and the corresponding portion of the given profile by L (Fig. 1).

The distance of the points of the profile L from the corresponding points of the profile L^0 along the outward normal to the latter, will be

* For certain conditions the problem of paper [1] has been solved by Devingtal' [3].

denoted by $\delta n(\sigma)$, where σ is any parameter which varies along the profile L^0. It is assumed that $\delta n(\sigma)$ is a sufficiently small quantity.

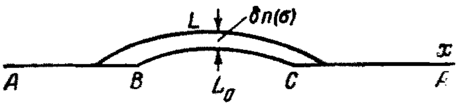

Fig. 1.

The stream function ψ^0, corresponding to flow past the profile L^0, satisfies in the hodograph plane the equation which to the first approximation coincides with Tricomi's equation:

$$\eta\psi^{\bullet}_{\theta\theta} + \psi^{\bullet}_{\eta\eta} = 0 \tag{1}$$

Here θ is the angle of inclination of the velocity, η is a known function of the modulus of the velocity, employed in the tneory of transonic flow.

Let us recall the formulation of Problem I in the hodograph plane (Figs. 2,3).

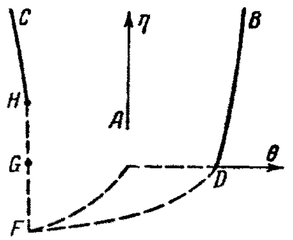

Fig. 2.

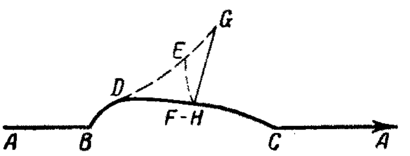

Fig. 3.

The boundary conditions are the following:

$$\psi^\circ = 0 \text{ on } HCABD, \tag{2}$$

$$\psi^\circ = k\rho^{-1/2}\sin\frac{1}{2}t + 0(1) \tag{3}$$

in a neighborhood of the point A, where

$$\rho = \sqrt{\theta^2 + \frac{4}{9}(\eta^{3/2} - \eta_A^{3/2})^2}$$

$$\left(\rho\sin t = \theta, \quad \rho\cos t = \frac{2}{3}\left[\eta^{3/2} - \eta_A^{3/2}\right]\right) \tag{4}$$

$$\psi^\circ(\theta_G, \eta) = \psi^\circ(\theta_G, -\eta), \qquad \psi_\theta{}^\circ(\theta_G, \eta) = 0 \tag{5}$$

on the vertical segment FGH (here FG corresponds to the front side and GH to the rear side of the straight compression shock)

$$\psi^\circ(\theta, 0) = f(\theta) \quad \text{on } ED \tag{6}$$

Here *EF* is a characteristic. On this curve no boundary conditions at all are given. The curve *DF*, on which $\psi^0 = 0$, is determined later after the solution of the boundary problem.

If it is assumed that the points *B*, *C* are points where the profile forks, so that the values of the velocity there are different from zero, whilst the values of η are finite, then the curve *HCABD* is finite.

In this case the problem has been solved by Devingtal' [3]. His solution was obtained under the conditions $A = B = C$ (Fig. 4).

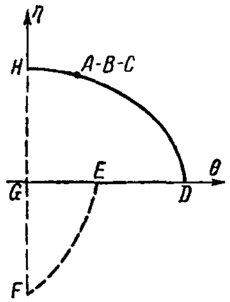

Fig. 4.

After the solution of Problem I, by means of the well known formulas of Chaplygin, we find the shape of the profile *BDFHC* in the physical plane, and also the straight compression shock *FGH*.

For the solution of Problem II it is convenient to make use of the perturbation in the modified stream function [4]

$$\delta\omega = \omega - \omega^\circ \tag{7}$$

where

$$\omega^\circ = \psi^\circ - \frac{\rho}{\rho_0}(uy^\circ - vx^\circ), \qquad \omega = \psi - \frac{\rho}{\rho_0}(uy - vx)$$

Here x, v, x^0, v^0, are the values of the Cartesian coordinates in the physical plane corresponding to θ, η Problems I and II, respectively.

The functions ω^0, ω and therefore also $\delta\omega$ satisfy the equation

$$\delta\omega_{\theta\theta} + \frac{\partial}{\partial\eta}\left(\frac{\delta\omega_\eta}{\eta}\right) = 0 \tag{8}$$

to the first approximation in the hodograph plane.

For small values of $\delta n(\sigma)$ the value of $\delta\omega$ is known on the curve *HCABDF* of the θ, η plane to an accuracy permitting only errors of the second order of smallness; we have [4]

$$\delta\omega = -\frac{\rho}{\rho_0} w\delta n \tag{9}$$

Furthermore, on the curve *BAC*, $\delta n = 0$, $\delta\omega = 0$.

At the point *A* we also have $\delta\omega = 0$. To establish the conditions on *FGH* we recall that the compression shock in Problem II is, generally speaking, no longer a straight line. Corresponding to it in the θ, η plane there is a certain curve $F'G'H'$, close to FGH. On this curve we have the conditions [2]

$$\psi_2 = \psi_1 \tag{10}$$

$$d\varphi_1 = \frac{C}{\sqrt{2}}\sqrt{-\eta_1 - \eta_2}\,d\psi_1, \qquad d\varphi_2 = \frac{C}{\sqrt{2}}\sqrt{-\eta_1 - \eta_2}\,d\psi_2 \qquad \left(C = \left(\frac{\varkappa+1}{2}\right)^{1/\varkappa-1}(\varkappa+1)^{1/3}\right) \tag{11}$$

where the indices 1, 2 correspond to the front and rear sides of the shock. Moreover, the quantities θ_1, θ_2, η_1, η_2 are connected by the equation

$$\theta_2 - \theta_1 = \frac{1}{\sqrt{2}}\sqrt{-\eta_1 - \eta_2}\,(\eta_2 - \eta_1) \tag{12}$$

We are considering the quantities $\theta_1 - \theta_G$, $\theta_2 - \theta_G$ to be small, and therefore, the quantity $\sqrt{-\eta_1 - \eta_2}$ is of the same order of smallness, whilst the quantity $(-\eta_1 - \eta_2)$ is of the second order of smallness. To the first approximation, condition (10) can be rewritten in the form

$$\psi^\circ(\theta_G, \eta) + \delta\psi(\theta_G, \eta) + \psi_\theta^\circ(\theta_G, \eta)\,\delta\theta_1 = \psi^\circ(\theta_G, -\eta) + \delta\psi(\theta_G, -\eta) + \psi_\theta^\circ(\theta_G, -\eta)\,\delta\theta_2$$

where $\delta\theta_1 = \theta_1 - \theta_G$ $\delta\theta_2 = \theta_2 - \theta_G$ (here we have used the fact that $(-\eta_1 - \eta_2)$ is a small quantity of the second order).

But by virtue of condition (5) it follows from this that

$$\delta\psi(\theta_G, \eta) = \delta\psi(\theta_G, -\eta) \tag{13}$$

where $\delta\psi$ and $\delta\omega$ are connected by the relation

$$\delta\psi = \delta\omega - \frac{\omega\delta\omega_\eta}{1 - M^2}$$

or, to the given approximation,

$$\delta\psi = \delta\omega + \frac{1}{2}(\varkappa+1)^{2/3}\frac{\omega\delta\omega_\eta}{\eta} \tag{14}$$

Accordingly, the function $\delta\omega(\theta_G, \eta)$ also has to be an even function of η:

$$\delta\omega(\theta_G, \eta) = \delta\omega(\theta_G, -\eta) \quad \text{on } FGH \tag{15}$$

The boundary conditions (9) and (15) determine the function $\delta\omega$. In order that the postulated boundary condition be permissible it is necessary that along the arc DF

$$-(-\eta)^{-1/2} < \frac{d\eta}{d\theta} \leqslant (-\eta)^{-1/2}$$

The theorem of uniqueness and existence for our flow pattern has been proved only for the equation of Lavrent'ev and Bitsadze under certain supplementary conditions imposed on the shape of the curve $HCABCDF$ (see below)

The curved compression shock of our flow pattern is determined by the basic equations (11) which, after the functions ϕ and ψ have been found, become differential equations for θ as a function of η in the lower and upper half-planes. In fact, we have

$$\varphi_\eta + \varphi_\theta \frac{d\theta}{d\eta} = \frac{C}{\sqrt{2}} \sqrt{-\eta_1 - \eta_2} \left(\psi_\eta + \psi_\theta \frac{d\theta}{d\eta} \right) \tag{16}$$

But

$$\frac{1}{\sqrt{2}} \sqrt{-\eta_1 - \eta_2} = \frac{\theta_2 - \theta_1}{\eta_2 - \eta_1} \approx \frac{\theta_2 - \theta_1}{2\,|\eta|} \tag{17}$$

where, neglecting errors of the second order of smallness, $\eta_2 = -\eta_1 = \eta$.

In formula (17) we have

$$\theta_2(\eta) = \theta(\eta), \qquad \theta_1(\eta) = \theta(-\eta) \qquad (\eta > 0) \tag{18}$$

Accordingly, we have, neglecting errors of the second order

$$\varphi_\eta(\theta_G, \eta) + \varphi_{\theta\eta}(\theta_G, \eta)\,\delta\theta_2 + \varphi_\theta(\theta_G, \eta) \frac{d\theta_2}{d\eta} = \frac{\theta_2(\eta) - \theta_1(\eta)}{2\eta} \psi_\eta(\theta_G, \eta) \tag{19}$$

$$-\varphi_\eta(\theta_G, -\eta) - \varphi_{\theta\eta}(\theta_G, -\eta)\,\delta\theta_1 - \varphi_\theta(\theta_G, -\eta) \frac{d\theta_1}{d\eta} = \frac{\theta_2(\eta) - \theta_1(\eta)}{2\eta} \psi_\eta(\theta_G, -\eta)$$

where $0 < \eta < \eta_H$. Moreover, the following relations must hold

$$\theta_1(0) = \theta_2(0), \qquad \theta(F') = \theta(H') \tag{20}$$

The proof of the existence and uniqueness of solution of Problem II will now be given for the equation of Lavrent'ev and Bitsadze

$$U_{xx} \operatorname{sign} y + U_{yy} = 0 \tag{21}$$

in the region *AOBCEA*, where *AOB* is a segment of the *y*-axis and *OA* = *OB*, the line *BC* is a straight line with slope $\pi/4$, and the arc *CEA* is symmetrical (Fig. 5) with respect to the bisector $v = x$. It can be assumed that $x_c = 1$.

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(24)

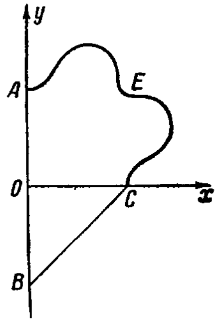

Fig. 5.

In the triangle *OBC* the solution has the form:

$$U = f(x - y) + g(x + y)$$

Here

$$g(2x - 1) = U(x, x - 1), \ f(1) = 0$$

The function g is therefore known on the basis of the boundary conditions. Accordingly, on the sector *OC* we have

$$U_x + U_y = 2g'(x)$$

As a result of the assumption concerning continuity of the first derivatives in crossing the segment *OC*, this equation holds good also for the approximation to the segment *OC* from above. Let us now map the region *OCA* conformally on the upper half of the ξ, η plane in such a way that points which are symmetrical relative to the bisector $y = x$ transform to points which are symmetrical relative to the η-axis and the origin of coordinates.

The boundary condition (24) takes the form:

$$U_\xi + U_\eta = 2g'(x)\left|\frac{dz}{d\zeta}\right| \qquad \begin{pmatrix} \zeta = \xi + i\eta \\ z = x + iy \end{pmatrix} \tag{25}$$

Let us introduce the complex variable $\Phi = U_\xi + iU_\eta$. Let $U_\xi(\xi, \eta) = \tau(\xi)$. Then in the upper half-plane we have [5]

$$\Phi(\zeta) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\tau(\xi')\,d\xi'}{\xi' - \zeta} + iC \tag{26}$$

If it is assumed that the boundary values of U on the curve *CA* have a bounded first derivative, then at infinity we have

$$\Phi(\infty) = (U_x + iU_y)\frac{dz}{d\zeta} = 0$$

Consequently, $C = 0$ and, finally,

$$\Phi(\zeta) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\tau(\xi')\,d\xi'}{\xi' - \zeta} \tag{27}$$

Let

$$\Phi = \Phi_1 + \Phi_2 \tag{28}$$

$$\Phi_1(\zeta) = \frac{1}{\pi i}\left(\int_{-\infty}^{-1}\frac{\tau(\xi')\,d\xi'}{\xi' - \zeta} + \int_{+1}^{+\infty}\frac{\tau(\xi')\,d\xi'}{\xi' - \zeta}\right) \tag{29}$$

$$\Phi_2(\zeta) = \frac{1}{\pi i}\int_{-1}^{+1}\frac{\tau(\xi')\,d\xi'}{\xi' - \zeta} \tag{30}$$

When ξ lies outside the interval $-1 < \xi < 1$ the function $\tau(\xi)$ is known as far as the boundary conditions, and consequently the function

$\Phi_1(\zeta)$ is known. We notice that when $-1 < \xi < 1$, we have $U_{1\xi}(\xi, 0) = 0$. When $-1 < \xi < 1$ we have $r(\xi) = -r(-\xi)$, so that according to (30)

$$U_{2\eta}(\xi, +0) = U_{2\eta}(-\xi, +0)$$

When ξ lies outside the interval $-1 < \xi < 1$ we have $U_{2\xi} = 0$.

Accordingly, for the determination of the function we obtain the following Riemann-Hilbert problem:

$$\begin{aligned}
&1)\quad U_{2\xi} = 0 \text{ when } \xi < -1 \text{ and } 1 < \xi \\
&2)\quad U_{2\xi} + U_{2\eta} = 2g'[x(\xi)]\left|\frac{dz}{d\zeta}\right| - U_{1\eta}(\xi, 0) \quad \text{when } 0 < \xi < 1 \\
&3)\quad U_{2\eta} - U_{2\xi} = 2g'[x(-\xi)]\left|\frac{dz}{d\zeta}\right|_{\zeta=-\xi} - U_{1\eta}(-\xi, 0) \quad \text{when } -1 < \xi < 0
\end{aligned} \tag{31}$$

The solution of this problem is determined in a unique manner as a result of the supplementary condition $\Phi_2(\infty) = 0$; this solution has the form:

$$\Phi_2(\zeta) = \frac{1+i}{2\pi i}\sqrt[4]{\frac{1-\zeta^2}{\zeta^2}}\int_{-1}^{+1}\frac{c(\xi')}{\xi'-\zeta}\sqrt[4]{\frac{\xi'^2}{1-\xi'^2}}\,d\xi \tag{32}$$

where

$$\begin{aligned}
c(\xi) &= 2g'[x(\xi)]\left|\frac{dz}{d\zeta}\right|_{\zeta=\xi} - U_{1\eta}(\xi, 0) \qquad (0 < \xi < 1) \\
c(\xi) &= -2g'[x(-\xi)]\left|\frac{dz}{d\zeta}\right|_{\zeta=-\xi} + U_{1\eta}(-\xi, 0) \qquad (-1 < \xi < 0)
\end{aligned} \tag{33}$$

By $[(1-\zeta^2)/\zeta^2]^{1/4}$ we denote that branch of this multi-valued function which assumes a real positive value as $\zeta \to \xi + i0$, $0 < \xi < 1$. In the same way the existence and uniqueness of the solution is demonstrated in the case under consideration.

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